

Space-Time as an Orderparameter Manifold in Random Networks and the Emergence of Physical Points

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Abstract

In the following we are going to describe how macroscopic space-time is supposed to emerge as an orderparameter manifold or superstructure floating in a stochastic discrete network structure. As in preceeding work (mentioned below), our analysis is based on the working philosophy that both physics and the corresponding mathematics have to be genuinely discrete on the primordial (Planck scale) level. This strategy is concretely implemented in the form of *cellular networks* and *random graphs*. One of our main themes is the development of the concept of *physical (proto)points* as densely entangled subcomplexes of the network and their respective web, establishing something like *(proto)causality*. It may perhaps be said that certain parts of our programme are realisations of some old and qualitative ideas of Menger and more recent ones sketched by Smolin a couple of years ago. We briefly indicate how this *two-story-concept of space-time* can be used to encode the (at least in our view) existing non-local aspects of quantum theory without violating macroscopic space-time causality!

1 Introduction

In a couple of recent papers ([1] to [4]) we began to develop some facets of an extensive programme we formulated there, i.e. rebuild ordinary continuum physics or mathematics as kind of a coarse grained limit of a much more primordial and genuinely discrete “*theory of everything*”, including, in particular, a discrete foundational theory of (proto) space-time as the universal receptacle or substratum of all physical processes.

A corresponding philosophy is presently held by a substantial minority of workers in the diversified field described a little bit vaguely by the catchword *quantum gravity* and we commented on some of the various approaches, at least as far as we are aware of them, in the foregoing papers. We therefore refer the interested reader to these papers as to references we do not mention in the following just for sake of brevity. As an exception we mention only the early and prophetic remarks made by Penrose in e.g. [5] about the surmised combinatorial substratum underlying our continuous space-time, the ideas of Smolin, sketched at the end of [6], because they are surprisingly close to our working philosophy and the work of ’t Hooft ([7]) which is based on the model system of *cellular automata* (the more rigid and regular relatives of our *dynamic cellular networks* introduced in the following).

Our personal working philosophy is that space-time at the very bottom (i.e. near or below the notorious Planck scale) resembles or can be modeled as an evolving information processing cellular network, consisting of elementary modules (with a, typically, simple internal discrete state space) interacting with each other via dynamical bonds which transfer the elementary pieces of information among the nodes. It is a crucial and perhaps characteristic extra ingredient of our framework that the bonds (i.e. the elementary interactions) are not simply dynamical degrees of freedom (as with the nodes their internal state space is assumed to be simple) but can a fortiori, depending on the state of the local network environment, be created or annihilated, i.e. can be temporarily dead or alive respectively active or inactive! This property opens the door to the wide field of *dimensional* or *geometrical phase transitions* and *selforganisation* of space-time as an *emergent* collective *order parameter field*, floating over a much more erratic and chaotic, but largely hidden, “underworld”.

In this context various fundamental questions of principle pose themselves both with respect to physics and the appropriate mathematical apparatus. In paper [1] we dealt primarily with dimensional concepts on such discrete and irregular spaces. It furthermore became apparent that there exist close ties to the theory of *fractal sets*. Papers [3] and [4], on the other hand, are, among other things, devoted to the development of several versions of *discrete analysis* and *discrete differential geometry* respectively *discrete functional analysis* with certain zones of contact with *noncommutative geometry*.

All these papers were to a large extent dominated by mathematical questions,

i.e. the development of the appropriate mathematical arsenal in order to establish something like *discrete analysis and geometry* on highly erratic spaces like our irregular (almost random) networks. In the following we want to deal with foundational problems of a (slightly) more physical character which, traditionally, belong more to the realm of *quantum gravity*.

As has been beautifully reviewed by Isham in various papers (see e.g. [8]) one could, among several possible attitudes, adopt the perhaps most radical working philosophy in quantum gravity and speculate that both *quantum theory* and *gravity* are merely secondary and derived effects or, expressed in more physical terms, so-called *effective theories* of an underlying more primordial and all embracing theory of a markedly *combinatorial* flavor. A theory comprising quantum theory and gravitation as *emergent* subtheories should, first of all, provide a framework in which the emergence of something we are used to call *space-time* respectively *quantum vacuum* can be expressed or discussed, most notably the emergence of the *continuum* from the *discrete* and the concept of physical *space-time points* and their intricate dynamical web. In other words, the main theme of this paper will be the description of the fine structure of the substratum underlying our continuum space-time and its dynamics together with an investigation of its propensity for *pattern creation*, that is, patterns which, we hope, will serve as the *protoforms* of the building blocks and concepts of our present day versions of continuum (quantum) physics and gravity.

Remark: We were recently kindly informed by El Naschie that a very early source where a couple of related ideas can be found (in a however heuristic form) is the contribution of Menger (perhaps better known from his research on topological dimension or fractal sets like the Menger sponge) in [9]. In this essay he entertains very interesting ideas about the necessity of a new *geometry of the microcosmos* based on the *geometry of lumps* and the concept of a *statistical metric space*. We note in passing that these are exactly the concepts we will develop in the following. Quite remarkable in this respect is also Einsteins thoughtful commentary at the end of the same volume about the *discrete* and the *continuum*.

In this context we also want to mention the *Cantorian space-time approach* of El Naschie et al. who tries to model microspace as a particular type of (*random*) *fractal* (see e.g. [10]). Another interesting early source with a possible bearing on our approach may also be v.Neumanns concept of ‘*continuum geometry*’ which is briefly described in [11]. As we became aware of these ideas only very recently, possible connections to our strand of ideas shall be analyzed elsewhere.

A perhaps not so obvious source of inspiration should also be mentioned here which, at first glance, is concerned with a seemingly different aspect of nature, i.e. the science of *complexity and emergent behavior* as it is understood by e.g. the *Santa Fe group* and which may be encoded in the metaphor “*complexity at the edge of chaos*” (a popular but beautiful account of this philosophy is [12];

for more details we recommend the bulletins issued by the institute, quite a few corresponding references can e.g. be found in [12]).

One of the reasons of our affinity to this philosophy is our belief that many of the processes and organizing principles which are acting behind the scene in, say, neural network science or the *coevolving fitness landscapes of complex adaptive systems* do have their (possibly rudimentary) counterparts already on the most primordial level of natural science i.e. fundamental physics. In a sense this is another version of ‘*grand unification*’.

As will be seen in the following, there exist even more surprising links to other seemingly remote areas of current research as we learnt very recently, a catchword being ‘*small-world networks*’ (see [13]). Some of the mechanisms being effective in our cellular network environment are expected to drive it towards a *phase transition threshold* where space-time as kind of a *superstructure* begins to emerge exhibiting features which are also observed in this so-called small-world scenario.

The paper is organised as follows. In the following section we define the conceptual context. In section 3 we introduce the fundamental framework of random graphs and prove a couple of useful mathematical theorems about them. In section 4 we briefly describe the presumed primordial initial phase of our network. Section 5 (together with section 3) is the central section of the paper. In it we make the physical concepts mathematically precise, i.e. map them on the corresponding mathematical objects, and derive a variety of (rigorous) results about the conjectured orderparameter manifold ST and its building blocks, the so-called ‘*physical points*’.

2 The Cellular Network Environment

In this section we will set the stage for the investigations to follow in the sections below. As was already done in [1] to [4] (to which the reader is referred for more motivation) we design the underlying substratum of our world or, more specifically, of our physical space-time vacuum as a *cellular network*, denoted by QX (quantum space; which is, however, at the moment only a metaphor), its elementary building blocks being *nodes* and *bonds*. As we have said in the introduction, the bond interactions are not only allowed to vary in strength but, a fortiori, can be switched off or on, depending on the state of their local environment. That is, stated in physical terms, bonds can be created or annihilated in the course of network evolution, which (hopefully) enables the system to undergo ‘*geometric phase transitions*’ being accompanied by an ‘*unfolding*’ and ‘*pattern formation*’, starting e.g. from a less structured chaotic initial phase. In other words and in contrast to, say, ‘*cellular automata*’, which are relatively rigid and regular in their wiring and geometric structure (in particular: with the bonds typically being non-dynamical), our cellular networks do not carry such a rigid overall order

as an external constraint (e.g. a regular lattice structure); their wiring is dynamical and thus behaves randomly to some extent. To put it briefly, order and modes of regularity are hoped to emerge via a process of ‘*self-organization*’ in a *background independent* way.

Remarks:

1. Some clarifying comments are perhaps in order at this point. The modelling of the depth structure of space-time as a cellular network consisting of nodes and bonds should not necessarily be understood in a plain bodily sense. One should rather consider it as a representation or emulation of the main characteristics of the physical scenario, as it is common practice in the art of model building being prevalent in modern physics. There may, in particular, exist a variety of superficially different systems, the logical structure of which can nevertheless be encoded in roughly the same abstract underlying network model. It is our personal belief that such a discrete network, governed by a relatively simple but cleverly chosen dynamical law, is capable of generating most if not all of the phenomena and emergent laws on which our ordinary continuum physics is usually grounded. That such a hypothesis is not entirely far-fetched may e.g. be inferred from the emerging complexity of such a simple cellular automaton model as the famous ‘*game of life*’.
2. A typical example is the *geometry of lumps* envisaged by Menger. Take as lumps the hypothetical ‘*infinitesimal*’ grains of space or space-time which cannot be further resolved (be it in a practical or principal sense). Let them overlap according to a certain rule so that they can interact or exchange information. Draw a node for each such lump and a bond for each two lumps which happen to overlap. In *combinatorial topology* such a *combinatorial complex* is called the *nerve* of the *set system* (cf. e.g. [14]). In a next step one may encode the respective strengths of interaction or degrees of overlap in a valuation of the corresponding bonds, yielding a cellular network of the kind we are having in mind. A fortiori one can make these mutual overlaps into dynamical variables, i.e. let them change in “time”.

A certain class of relatively simple cellular networks is the following.

Definition 2.1 (Class of Cellular Networks)

1. “Geometrically” our networks represent at each fixed ‘clock time’ are ‘labeled graphs’, i.e. they consist of nodes $\{n_i\}$ and bonds $\{b_{ik}\}$, with the bond b_{ik} connecting the nodes (cells) n_i, n_k . We assume that the graph has neither elementary loops nor multi-bonds, that is, only nodes with $i \neq k$ are connected by at most one bond.

2. At each site n_i we have a local node state $s_i \in q \cdot \mathbb{Z}$ with q , for the time being, a certain not further specified elementary quantum. The bond variables J_{ik} , attached to b_{ik} , are in the most simplest cases assumed to be two- or three-valued, i.e. $J_{ik} \in \{\pm 1\}$ or $J_{ik} \in \{\pm 1, 0\}$
3. As to the (time (in)dependent) distribution of bonds over the graph there exist several (physically motivated) possibilities as will be explained in the following. In defining the mathematical model one may choose to start with an a priori fixed set of bonds (i.e. independent of e.g. ‘clock time’ or something else) and shift the corresponding dynamics to the bond variables J_{ik} . Physically one may, on the other hand, prefer to make this set of bonds itself a dynamical set, i.e. with bonds vanishing and emerging. Both schemes have their respective advantages.

Remarks

1. In the proper graph context the notions ‘vertex’ and ‘edge’ are perhaps more common (see e.g. [15]). As to other concepts occurring in graph theory see below.
2. This is, in some sense, the simplest choice one can make. It is an easy matter to employ instead more complicated or reduced internal state spaces like, say, groups, manifolds etc. One could in particular replace \mathbb{Z} by one of its quotient groups as e.g. \mathbb{Z}_n or impose suitable boundary conditions.
3. It is useful to give the bonds b_{ik} an ‘orientation’, i.e. (understood in an precise algebraic/geometric sense) $b_{ik} = -b_{ki}$. This implies the compatibility conditions $J_{ik} = -J_{ki}$.

In a next step we impose a dynamical law on our model network. In this respect we are of course inspired by ‘cellular automaton laws’ (see e.g. [16]). The main difference is however that in our context also the bonds are dynamical degrees of freedom and that, a fortiori, they can become dead or alive (active or inactive), so that the whole net is capable of performing drastic topological/geometrical changes in the course of clock time.

A large class of dynamical ‘local laws’ are e.g. the following ones: We assume that all the nodes/bonds at discrete ‘(clock) time’ $t + \tau$, τ an elementary clock time step (i.e. $t = z \cdot \tau$ with $z \in \mathbb{Z}$), are updated according to a certain local rule which relates the internal node space/bond space of each node/bond at time $t + \tau$ with the states of the nodes/bonds of a certain fixed local neighborhood at time t .

It is important that, generically, such a law does not lead to a reversible time evolution, i.e. there will typically exist attractors in total phase space (the overall

configuration space of the node and bond states). On the other hand, there exist strategies to develop reversible network laws.

A crucial ingredient of our network laws is what we would like to call a ‘*hysteresis interval*’. We will assume that our network, QX , starts from a densely entangled ‘*initial phase*’ QX_0 , in which practically every pair of nodes is on average (to be understood in a statistical sense) connected by an ‘*active*’ bond, i.e. $J_{ik} = \pm 1$. Our dynamical law will have a built-in mechanism which switches bonds off (more properly: sets $J_{ik} = 0$) if local fluctuations among the node states become too large. Then there is a certain hope that this mechanism may trigger an ‘*unfolding phase transition*’, starting from a local seed of spontaneous large fluctuations towards a new phase (an attractor) carrying an emergent ‘*super structure (proto space-time)*’, which we want to identify with the hidden discrete substratum of our ordinary continuous space-time.

One example of such a law is given in the following definition.

Definition 2.2 (Example of a Local Law) *At each clock time step a certain ‘quantum’ q is exchanged between, say, the nodes n_i, n_k , connected by the bond b_{ik} such that*

$$s_i(t + \tau) - s_i(t) = q \cdot \sum_k J_{ki}(t) \quad (1)$$

(i.e. if $J_{ki} = +1$ a quantum q flows from n_k to n_i etc.)

The second part of the law describes the back reaction on the bonds (and is, typically, more subtle). This is the place where the so-called ‘hysteresis interval’ enters the stage. We assume the existence of two ‘critical parameters’ $0 \leq \lambda_1 \leq \lambda_2$ with:

$$J_{ik}(t + \tau) = 0 \quad \text{if} \quad |s_i(t) - s_k(t)| =: |s_{ik}(t)| > \lambda_2 \quad (2)$$

$$J_{ik}(t + \tau) = \pm 1 \quad \text{if} \quad 0 < \pm s_{ik}(t) < \lambda_1 \quad (3)$$

with the special proviso that

$$J_{ik}(t + \tau) = J_{ik}(t) \quad \text{if} \quad s_{ik}(t) = 0 \quad (4)$$

On the other side

$$J_{ik}(t + \tau) = \begin{cases} \pm 1 & J_{ik}(t) \neq 0 \\ 0 & J_{ik}(t) = 0 \end{cases} \quad \text{if} \quad \lambda_1 \leq \pm s_{ik}(t) \leq \lambda_2 \quad (5)$$

In other words, bonds are switched off if local spatial charge fluctuations are too large, switched on again if they are too small, their orientation following the sign of local charge differences, or remain inactive.

Remarks

1. We do not choose the “current” $q \cdot J_{ik}$ proportional to the “voltage difference” $(s_i - s_k)$ as e.g. in Ohm’s law since we favor a *non-linear* network which is capable of *self-excitation* and *self-organization* rather than *self-regulation* around a relatively uninteresting equilibrium state! The balance between dissipation and amplification of spontaneous fluctuations has however to be carefully chosen (“*complexity at the edge of chaos*”)
2. We have emulated these local network laws on a computer and made extensive numerical studies of a lot of network or graph characteristics (which will be published elsewhere). It is not yet clear whether the above simple network law does already everything we are expecting. We studied networks up to roughly five thousand nodes. This is still quite a small number if one intends to simulate the whole universe where node numbers of 10^{100} or so should be envisaged. Furthermore, the investigation of certain complex graph properties is almost *non-polynomially complete*. In any case, it is fascinating to observe the enormous capability of such intelligent networks to find attractors very rapidly, given the enormous accessible phase space (as to this particular and important feature confer the remarks of S.Kauffman in [12])
3. In the above class of laws a direct bond-bond interaction is not yet implemented. We are prepared to incorporate such a (possibly important) contribution in a next step if it turns out to be necessary. In any case there are not so many ways to do this in a sensible way. Stated differently, the class of possible physically sensible interactions is perhaps not so numerous.
4. Note that – in contrast to e.g. Euclidean lattice field theory – the so-called ‘*clock time*’ t is, for the time being, not standing on the same footing as potential “coordinates” in the network (e.g. curves of nodes/bonds). We rather expect so-called ‘*physical time*’ to emerge as sort of a secondary collective variable in the network, i.e. to be different from the clock time (while being of course functionally related to it).

We nevertheless regard Remark 4 to be consistent with the spirit of relativity. What Einstein was really teaching us is that there is a (dynamical) interdependence between what we experience as space respectively time, not that they are absolutely identical! In any case, the assumption of an overall clock time is at the moment only made just for convenience in order to make the model system not too complicated. If our understanding of the complex behavior of the network dynamics increases, this assumption may be weakened in favor of a possibly local and/or dynamical clock frequency. A similar attitude should be adopted concerning concepts like ‘*Lorentz-(in)covariance*’ which we also consider as ‘*emergent*’ properties. It is needless to say that it is of tantamount importance to understand

the way how these patterns do emerge from the relatively chaotic background, questions which will be addressed in future work (as to some related ideas about such issues see also [17] or [18]).

The above example of a network law is only one candidate from a whole class of such laws. For one, it is quite evident that the ‘*local state spaces*’ living over the respective nodes and bonds can be chosen in a more general way. For another, the local dynamical law can be chosen more general.

Definition 2.3 (General Local Law on Cellular Networks) *Each node n_i can be in a number of internal states $s_i \in \mathcal{S}$. Each bond b_{ik} carries a corresponding bond state $J_{ik} \in \mathcal{J}$. Then the following general transition law is assumed to hold:*

$$s_i(t + \tau) = ll_s(\{s_k(t)\}, \{J_{kl}(t)\}) \quad (6)$$

$$J_{ik}(t + \tau) = ll_J(\{s_l(t)\}, \{J_{lm}(t)\}) \quad (7)$$

$$(\underline{S}, \underline{J})(t + \tau) = LL((\underline{S}, \underline{J})(t)) \quad (8)$$

where ll_s , ll_J are two maps (being the same over the whole graph) from the state space of a local neighborhood of the node or bond on the lhs to \mathcal{S}, \mathcal{J} , yielding the updated values of s_i and J_{ik} . \underline{S} and \underline{J} denote the global states of the nodes and bonds and LL the global law built from the local laws at each node or bond.

We close this section with a brief résumé of the characteristics an interesting network dynamics should encode.

Résumé 2.4 *Irrespectively of the technical details of the dynamical evolution law under discussion it should emulate the following, in our view crucial, principles, in order to match certain fundamental requirements concerning the capability of ‘emergent’ and ‘complex’ behavior.*

1. *As is the case with, say, gauge theory or general relativity, our evolution law on the surmised primordial level should implement the mutual interaction of two fundamental substructures, put sloppily: “geometry” acting on “matter” and vice versa, where in our context “geometry” is assumed to correspond in a loose sense with the local and/or global bond states and “matter” with the structure of the node states.*
2. *By the same token the alluded selfreferential dynamical circuitry of mutual interactions is expected to favor a kind of ‘undulating behavior’ or ‘self-excitation’ above a return to some uninteresting ‘equilibrium state’ as is frequently the case in systems consisting of a single component which directly acts back on itself. This propensity for the ‘autonomous’ generation of undulation patterns is in our view an essential prerequisite for some form of “protoquantum behavior” we hope to recover on some coarse grained and less primordial level of the network dynamics.*

3. *In the same sense we expect the overall pattern of switched-on and -off bonds to generate a kind of “protogravity”.*

Remark: As in the definition of evolution laws of ‘*spin networks*’ by e.g. Markopoulou, Smolin and Borissov (see [19] or [20]), there are in our case more or less two possibilities: treating evolution laws within an integrated space-time formalism or regard the network as representing space alone with the time evolution being implanted via some extra principle (which is the way we have chosen above). As the interrelation of these various approaches and frameworks is both very interesting and presently far from obvious we plan to compare them elsewhere.

3 The Cellular Network as a (Random) Graph

There are many different aspects one can study as regards our class of complex cellular networks. As in a purely geometric sense they are *graphs* it is, in a first step, a sensible strategy to suppress all the other features like e.g. the details of the internal state spaces of nodes and bonds and concentrate instead on its pure ‘*wiring diagram*’ and its reduced dynamics. This is already an interesting characteristic of the network (perhaps somewhat reminiscent of the ‘*Poincaré map*’ in the theory of chaotic systems) as bonds can be switched on and off in the course of time so that already the wiring diagram will constantly change. Furthermore, as we will see, it encodes the complete *near-* and *far-order structure* of the network, that is, it tells us which regions are experienced as near by or far apart (in a couple of possible physical ways such as strength of correlations, with respect to some physical metric or ‘*statistical distance*’). Evidently this is one of the crucial features we expect from something like physical space-time.

We start with the introduction of some graph theoretical definitions.

Definition 3.1 (Simple Locally Finite (Un)directed Graph))

1. *We write the ‘simple’ ‘labeled’ graph as $G := (V, E)$ where V is the countable set of nodes $\{n_i\}$ (or vertices) and E the set of bonds (edges). The graph is called simple if there do not exist elementary ‘loops’ and ‘multiple edges’, in other words: each existing bond connects two different nodes and there exists at most one bond between two nodes. (We could of course also discuss more general graphs). Furthermore, for simplicity, we assume the graph to be connected, i.e. two arbitrary nodes can be connected by a sequence of consecutive bonds called an edge sequence or ‘walk’. A minimal edge sequence, that is one with each intermediate node occurring only once, is called a ‘path’ (note that these definitions may change from author to author).*
2. *We assume the graph to be ‘locally finite’ (but this not always really necessary), that is, each node is incident with only a finite number of bonds.*

Sometimes it is useful to make the stronger assumption that this ‘vertex degree’, v_i , (number of bonds being incident with n_i), is globally bounded away from ∞ .

Observation/Definition 3.2 *Among the paths connecting two arbitrary nodes there exists at least one with minimal length. This length we denote by $d(n_i, n_k)$. This d has the properties of a metric, i.e:*

$$d(n_i, n_i) = 0 \quad (9)$$

$$d(n_i, n_k) = d(n_k, n_i) \quad (10)$$

$$d(n_i, n_l) \leq d(n_i, n_k) + d(n_k, n_l) \quad (11)$$

(The proof is more or less evident).

Corollary 3.3 *With the help of the metric one gets a natural neighborhood structure around any given node, where $\mathcal{U}_m(n_i)$ comprises all the nodes, n_k , with $d(n_i, n_k) \leq m$, $\partial\mathcal{U}_m(n_i)$ the nodes with $d(n_i, n_k) = m$.*

Remark: The restriction to connected graphs is, for the time being, only made for convenience. If one wants to study geometric phase transitions of a more fragmented type, it is easy to include these more general types of graphs. In the context of *random graphs* (which we will introduce below) one can even derive probabilistic criteria concerning geometric properties like connectedness etc.

Observation 3.4 (Projected Graph Process) *The projection of the full network process on the space of corresponding graphs over, say, a given fixed set of nodes defines what one may call a projected graph process with respect to clocktime t . The reduced dynamics or evolution of the graph, $G(t)$, i.e. the wiring diagram of the network, is then expressed by the creation and annihilation of bonds during consecutive time steps.*

Note however that this type of process is, in general, not an intrinsic one. The evolution law on the graph level is only defined extrinsically via the network law and generically cannot be reduced to a law acting on graph space as the projection is accompanied by an information loss. That is, different network states may typically project on the same graph state and, a fortiori, will lead in most cases after another time step even to different graph states.

The graphs or networks we are actually interested in are expected to be extremely large. According to our philosophy they are to emulate the full physical vacuum together with all its more or less macroscopic excitations, or, in other words, the entire evolving universe. Furthermore the assumed *clocktime interval* τ is extremely short (in fact of Plancktime order). On the other side, it is part of our working philosophy that the phenomena we are observing in e.g. present day high energy physics and, a fortiori, macroscopic physics are of the nature of

collective (frequently large scale) excitations of this medium both with respect to space and time (in Planck units). In other words, each of these patterns is expected to contain, typically, a huge amount of nodes and bonds and to stretch over a large number of clocktime intervals. This then suggests the following approach which has been fruitful again and again in modern physics.

The Statistical Hypothesis 3.5 *Following the above arguments it makes sense to study so-called ‘graph properties’ within a certain statistical framework to be further explained below.*

There are some remarks in order as this concept has at least two, to some extent different, facets.

Remarks:

1. First, there exists the usual strategy to average over small scale events or degrees of freedom in order to acquire some (more or less) coarse grained but, on the other side, more generic information about the evolving network or graph (examples being given below).
2. Second, while our network is, on the one side, up to now completely deterministic, one may nevertheless entertain the idea that, irrespective of the details of potential initial conditions or of our network law, there exist at least whole *classes* of them which lead to the same gross phenomena (whereas the underlying microstates at each given time step may be quite distinct). It hence seems to be advisable to group appropriate sets of microstates into entities of a slightly higher order of organisation called *phases*. A typical *initial phase* will be described below.
3. A characteristic phenomenon in this context is the following. Irrespective of the huge accessible *microscopic phase space*, networks like ours typically manage to reach an *attractor* (which may of course depend on the initial phase) after a frequently surprisingly short *transient* which makes the assumed *robustness* against the details of initial conditions quite apparent. Similar strategies have been adopted in the investigation of cellular automata (see e.g. [21] or [22]).
4. Another strand of reasoning is based on a slightly different bundle of ideas. Consider all possible graphs, $\{G_i\}$, over n vertices or nodes. In contrast to the above approach, which deals rather with ‘*practical statistics*’, we will now construct a *probability space* in at least two ways.

The Random Graph Idea 3.6 *Take all possible labeled graphs over n nodes as probability space \mathcal{G} (i.e. each graph represents an elementary event). The*

maximal possible number of bonds is $N := \binom{n}{2}$, which corresponds to the unique simplex graph (denoted usually by K_n). Give each bond the independent probability $0 \leq p \leq 1$, (more precisely, p is the probability that there is a bond between the two nodes under discussion). Let G_m be a graph over the above vertex set having m bonds. Its probability is then

$$\text{pr}(G_m) = p^m \cdot q^{N-m} \quad (12)$$

where $q := 1 - p$. There exist $\binom{N}{m}$ different labeled m -graphs G_m and the above probability is correctly normalized, i.e.

$$\text{pr}(\mathcal{G}) = \sum_{m=0}^N \binom{N}{m} p^m q^{N-m} = (p + q)^N = 1 \quad (13)$$

This probability space is sometimes called the space of ‘binomially random graphs’ and denoted by $\mathcal{G}(n, p)$. Note that the number of edges is binomially distributed, i.e.

$$\text{pr}(m) = \binom{N}{m} p^m q^{N-m} \quad (14)$$

and

$$\langle m \rangle = \sum m \cdot \text{pr}(m) = N \cdot p \quad (15)$$

Proof of the latter statement:

$$\langle m \rangle = d/d\lambda|_{\lambda=1} \left(\sum \binom{N}{m} (\lambda p)^m q^{N-m} \right) = d/d\lambda|_{\lambda=1} (\lambda p + q)^N = Np \quad (16)$$

or, with e_i being the independent ‘Bernoulli $(0, 1)$ -variables’ (as to this notion cf. e.g. [23]) belonging to the bonds:

$$\langle e_i \rangle = p \quad \text{hence} \quad \langle m \rangle = \sum_1^N \langle e_i \rangle = Np \quad (17)$$

□

(The use of the above Bernoulli variables leads also to some conceptual clarifications in other calculations).

A Variant Approach 3.7 A slightly different probability space can be constructed by considering only graphs with a fixed number, m , of bonds and give each the probability $\binom{N}{m}^{-1}$ as there are exactly $\binom{N}{m}$ of them. The corresponding probability space, $\mathcal{G}(n, m)$, is called the space of ‘uniform random graphs’.

The latter version is perhaps a little bit more common in pure mathematics as this concept was introduced mainly for purely combinatorial reasons which have nothing to do with our own strand of ideas. The whole theory was rather developed by Erdős and Rényi in the late fifties and early sixties to cope with certain notorious (existence) problems in graph theory (for more information see e.g. [24] and [26], brief but concise accounts can also be found in chapt.VII of [15] or [27]).

Observation 3.8 *The two random graph models behave similarly if $m \approx p \cdot N$. Note however, that there exists a subtle difference between the two models anyway. In the former model all elementary bond random variables are independent while in the latter case they are (typically weakly) dependent.*

(While being plausible this statement needs a proof which can be found in e.g. [24]).

The really fundamental observation made already by Erdős and Rényi (a rigorous proof of this deep result can e.g. be found in [26]) is that there are what physicists would call *phase transitions* in these *random graphs*. To go a little bit more into the details we have to introduce some more graph concepts.

Definition 3.9 (Graph Properties) *Graph properties are certain particular random variables (indicator functions of so-called events) on the above probability space \mathcal{G} . I.e., a graph property, Q , is represented by the subset of graphs of the sample space having the property under discussion. To give some examples: i) connectedness of the graph, ii) existence and number of certain particular subgraphs (such as subsimplices etc.), iii) other geometric or topological graph properties etc.*

Remark: In addition to that there are other more general random variables (‘graph characteristics’) describing the fine structure of graphs, some of which we will introduce below.

In this context Erdős and Rényi made the following important observation.

Observation 3.10 (Threshold Function) *A large class of graph properties (e.g. the ‘monotone increasing ones’, cf. the above cited literature) have a so-called ‘threshold function’, $m^*(n)$, so that for $n \rightarrow \infty$ the graphs under discussion have property Q almost surely for $m(n) > m^*(n)$ and almost surely not for $m(n) < m^*(n)$ or vice versa (more precisely: for $m(n)/m^*(n) \rightarrow \infty$ or 0; for the details see the above literature). The above version applies to the second kind of graph probability space, $\mathcal{G}(n, m)$. A corresponding result holds for $\mathcal{G}(n, p)$ with $p(n)$ replacing $m(n)$. That is, by turning on the probability p , one can drive the graph one is interested in beyond the phase transition threshold belonging to the graph property under study. Note that, by definition, threshold functions are only unique up to “factorization”, i.e. $m_2^*(n) = O(m_1^*(n))$ is also a threshold function.*

Example 3.11 (Connectedness) *The threshold function for the graph property connectedness is*

$$m^*(n) = n/2 \cdot \log(n) \text{ respectively } p^*(n) = \log(n)/n \quad (18)$$

Note that with the help of the above observation, i.e. for $m \approx p \cdot \binom{n}{2}$, we have for n large: $\binom{n}{2} \approx n^2/2$ and hence $p \cdot n^2/2 \approx n \log(n)$, i.e. $p \approx \log(n)/n$

Observation/Definition 3.12 (Combinatorial Graph Process) *It is perhaps conceptually more intuitive to envisage the passage of a random graph through its various epochs in a more dynamical way by introducing a combinatorial graph process, starting from the empty graph and adding at each discrete “time step” one edge, chosen at random. Each element \tilde{G} consists of a sequence of random graphs $\{G_m\}_0^N$. Note however that no real dynamics in e.g. a physical sense is implied. Therefore this kind of combinatorial process should not be confused with the real dynamics and evolution of our networks and graphs.*

In the following our main thrust will go towards the development of the concept of ‘proto space-time’ as an ‘order parameter manifold’ or ‘superstructure’ floating in our network QX and the concept of ‘physical points’ together with their respective ‘near- and far-order’. We therefore consider it worthwhile to illustrate the method of random graphs, graph properties and graph characteristics by applying it to a particular feature being of importance in the sequel.

Definition 3.13 (Subsimplices and Cliques) *With G a given fixed graph and V_i a subset of its vertex set V , the corresponding induced subgraph over V_i is called a subsimplex (ss), S_i , or complete subgraph, if all its nodes are connected by a bond. In this class there exist certain maximal subsimplices (mss), that is, every addition of another node destroys this property. These mss are called cliques in combinatorics.*

For reasons to be explained below we are extremely interested in such mss and their behavior (which depends of course on the given graph G under discussion). It is sometimes a certain art to design an appropriate random variable implementing a particular graph property and work out the necessary combinatorial probabilities. In our specific example we proceed as follows: We consider all possible graphs, G , over the fixed vertex set V of n nodes. For each subset $V_i \subset V$ of order r (i.e. number of elements) we define the following random variable:

$$X_i(G) := \begin{cases} 1 & \text{if } G_i \text{ is an } r\text{-simplex,} \\ 0 & \text{else} \end{cases} \quad (19)$$

where G_i is the corresponding induced subgraph over V_i with respect to $G \in \mathcal{G}$ (the probability space). Another random variable is then the *number of r -simplices in G* , denoted by $Y_r(G)$ and we have:

$$Y_r = \sum_{i=1}^{\binom{n}{r}} X_i \quad (20)$$

with $\binom{n}{r}$ the number of r -subsets $V_i \subset V$. With respect to the probability measure introduced above we then have for the *expectation values*:

$$\langle Y_r \rangle = \sum_i \langle X_i \rangle \quad (21)$$

and

$$\langle X_i \rangle = \sum_{G \in \mathcal{G}} X_i(G) \cdot pr(G_i = r\text{-simplex in } G). \quad (22)$$

With the sum running over all $G \in \mathcal{G}$ and X_i being one or zero we get

$$\langle X_i \rangle = pr(G_i \text{ an } r\text{-simplex, } G - E_i \text{ an arbitrary graph}) \quad (23)$$

where $G - E_i$ is the remaining graph after all the edges belonging to G_i have been eliminated. This yields

$$\langle X_i \rangle = p^{\binom{r}{2}} \cdot \sum_{G' \in \mathcal{G}'} pr(G') \quad (24)$$

where \mathcal{G}' is the probability space of graphs over V with all the bonds E_i being omitted. The maximal possible number of bonds belonging to \mathcal{G}' is

$$|E'| = \binom{n}{2} - \binom{r}{2}. \quad (25)$$

Each of these bonds can be on or off with probability p or $(1-p)$. To each graph of \mathcal{G}' belongs a unique labeled sequence of p 's and q 's and every such sequence does occur (i.e. with either p or q at label i). We hence have

$$\sum_{G'} pr(G') = (p+q)^{|E'|} = 1^{|E'|} = 1 \quad (26)$$

and we get

$$\langle X_i \rangle = p^{\binom{r}{2}} \quad (27)$$

There are now exactly $\binom{n}{r}$ possible r -subsimplices over the node set V , hence we arrive at the following important conclusion:

Conclusion 3.14 (Subsimplices) *The expectation value of the random variable number of r -subsimplices is*

$$\langle Y_r \rangle = \binom{n}{r} \cdot p^{\binom{r}{2}} \quad (28)$$

It is remarkable, both physically and combinatorially, that this quantity as a function of r , i.e. the order of subsimplices, has quite a peculiar numerical behavior. We are, among other things, interested in the typical order of cliques (where typical is understood in a probabilistic sense).

Observation/Definition 3.15 (Clique Number) *The maximal order of a clique contained in G is called its clique number, $cl(G)$. It is another random variable on the probability space $\mathcal{G}(n, p)$.*

An analysis of the above expression $\langle Y_r \rangle = \binom{n}{r} \cdot p^{\binom{r}{2}}$ as a function of r shows that it is typically very large (for n sufficiently large) for all $r \leq r_0$ and drops rapidly to zero for $r > r_0$ for r_0 some *critical value* to be determined.

Conclusion 3.16 *From the above one can infer that this value r_0 is an approximation for the above defined clique number of a typical random graph, depending only on p and n . In other words, it approximates the order of the largest occurring clique in the respective graph. A good approximation of r_0 is*

$$r_0 \approx 2 \log(n) / \log(p^{-1}) + O(\log \log(n)) \quad (29)$$

(cf. chapt. XI.1 of [24]).

Example 3.17 $(n = 100, p = 1/2) \Rightarrow r_0 \approx 12$

r_0 is an approximation of the typical order of the highest occurring clique in a random graph (understood in a probabilistic sense). For our investigation of the fine structure of space-time it is however of tantamount importance to acquire more detailed information about the typical number and distribution of cliques in a random graph for varying r (note that the above formula provides, up to now, only an estimate of the number of r -simplices). It seems to be a much more ambitious task to deal with the corresponding case in the extremal situation. This problem can however be solved by a slightly tricky approach.

The probability that the induced subgraph over r arbitrary but fixed vertices, n_1, \dots, n_r , (with the graph G given), is a ss is $p^{\binom{r}{2}}$ (see above). If it is, a fortiori, a clique or mss , this means that the addition of a single further vertex would destroy the property being a ss . In other words, each of the vertices in the graph G , different from the above n_1, \dots, n_r , is connected with the above vertex set, V_r , by fewer than r bonds. This can now be quantitatively encoded as follows:

Observation 3.18 (distribution of mss or cliques) *The probability that the induced subgraph G_r over r arbitrarily chosen vertices is already a mss is the product of the independent probabilities that it is a ss and that each of the remaining $(n - r)$ vertices is connected with V_r by fewer than r bonds. The latter probability is $(1 - p^r)^{n-r}$, hence:*

$$\text{pr}(G_r \text{ is a clique}) = (1 - p^r)^{n-r} \cdot p^{\binom{r}{2}} \quad (30)$$

From this we can immediately infer for Z_r , the number of r -cliques in the random graph, the following relation

$$\langle Z_r \rangle = \binom{n}{r} \cdot (1 - p^r)^{n-r} \cdot p^{\binom{r}{2}} \quad (31)$$

Conclusion 3.19 *By an estimation of the variance of Z_r we can conclude that the typical orders of occurring cliques lie between $r_0(n)/2$ and $r_0(n)$.*

Remark 3.20 *To make the above reasoning completely rigorous, it is again helpful to exploit the properties of the elementary $(0, 1)$ -edge-variables e_i . The probability that r arbitrarily selected bonds exist in the random graph is $\text{pr}(e_1 = \dots = e_r = 1) = p^r$, the complimentary possibility (i.e., that some of these bonds are missing), has hence the probability $(1 - p^r)$.*

Résumé 3.21 (Random Graph Approach) *There is of course no absolute guarantee that our network, following a deterministic evolution law and typically reaching after a certain transient one of possibly several attractors, can in every respect be regarded as the evolution of true random graphs. In other words, its behavior may not be entirely random. Quite to the contrary, we expect a shrinking of phase space during its evolution which manifests itself on a more macroscopic level as pattern creation.*

The underlying strategy is rather the following. At each clock time step, $G(t)$ is a graph having a definite number of bonds, say m . Surmising that $G(t)$ is sufficiently generic or typical we can consider it as a typical member of the corresponding family $\mathcal{G}(n, m)$ or $\mathcal{G}(n, p)$. Via this line of inference we are quite confident of being able to get some clues concerning the qualitative behavior of our network. As to this working philosophy the situation is not completely different from the state of affairs in many areas of, say, ordinary statistical physics. But nevertheless, a more detailed analysis (underpinned by concrete examples) of the validity of this ansatz would be desirable and shall be given in forthcoming work.

What we said above is on the one side corroborated by our own extensive computer simulations (performed by our former student Th. Nowotny). On the

other side such a possibly characteristic numerical deviation between theoretical results based on certain *a priori* statistical assumptions and computer simulations of concrete models was also found in [25] by Antonsen, who devised an approach which is different from ours in several respects but is similar in the general spirit.

4 A Qualitative Description of the Presumed Initial Phase

The central theme of this paper is the description and analysis of a certain superstructure, ST , emerging within our network QX as a consequence of a process which can be interpreted as a geometrical phase transition. In this picture ST , which we experience as space-time, plays the role of an order parameter manifold. Its emergence signals the transition from a *disordered* and *chaotic* initial phase to a phase developing a *near-/far-order*, i.e. a *causal structure*, and stable ‘*physical points*’ or ‘*lumps*’ (Menger).

As explained above, it is advantageous at various intermediate steps to coarse grain the amount of information being buried in the microstates of the network and concentrate rather on the geometric information encoded, say, in its pure wiring diagram, i.e. the corresponding graph process $G(t)$. Furthermore, as is the typical case with dynamical systems of such an extreme complexity, we are typically interested in the generic behavior and patterns and not so much in every, possibly only contingent, detail of the microstates of the network. That means, the natural object of our interest is what we called in section 3 a *phase*, that is, a whole class of microstates behaving similarly in a certain “macroscopic” or large-scale respect (yet to be defined). This is in complete analogy with the ideas of ordinary statistical physics and hence calls for a statistical approach and analysis of the kind we mentioned in the preceding sections.

In a first step we have to describe the presumed ‘*initial phase*’ of our network. This is the phase which, due to its chaotic behavior, cannot yet support a superstructure like ST and which is, a fortiori, assumed to be free of any other stable type of pattern. After some time of model building and experimenting emerged as the, in our view, most natural and plausible candidate a phase where, on average, almost every node is connected with any other node so that, among other things, all the nodes are roughly staying on equal footing with respect to each other. On the other side, the expected inherent wild fluctuations in the network entail that bonds happen to be permanently created and annihilated.

Remarks: There exist several reasons for selecting this scenario as initial phase.

1. For one, we want to emulate the transition of the universe from chaos to *complex* order. We expect that a phase in which practically all the nodes

and bonds are standing on an equal footing with respect to each other and in which “everything interacts with everything” in a highly erratic manner (given the presumed extremely high vertex degree) guarantees the complete absence of stable patterns or order of any other kind.

2. For another, our approach (as kind of a “*theory of everything*”) shall also be capable of describing the “*big bang era*” of our universe, i.e. the unfolding of space-time, starting from a tiny nucleus. It will turn out that such a scenario can in fact be described in a very natural and “*unforced*” way within our network framework which is of course very satisfying.
3. Lastly, such an initial phase is, as far as we can see, almost the only one which is free of contingent or adhoc assumptions like
e.g. a particular choice of the initial distribution of active bonds etc.

Assumption 4.1 (Initial Conditions) *We assume the initial phase of our network, dubbed henceforth QX_0 , to be generically an ‘almost complete graph’, i.e. the average number of active bonds is assumed to be almost maximal (in other words, QX_0 is almost a simplex). That implies that the number of active bonds is supposed to be “near” $\binom{|V|}{2}$ (with $|V|$ the number of nodes).*

Equivalently, we assume the average node degree in the phase QX_0 to remain near its maximum, i.e. $|V| - 1$.

Definition 4.2 (Notations and Conventions) *Averages can be taken over the network (or graph) at fixed clock time or with respect to clock time at e.g. a fixed node or both “space” and “time”. We abbreviate the corresponding averaging by*

$$\langle \cdot \rangle_s, \langle \cdot \rangle_t, \langle \cdot \rangle_{st} \tag{32}$$

Warning: This kind of averaging should not be confused with the expectation values defined within the random graph approach (while, according to our working philosophy, there should be various interrelations).

Note that, as in cellular automata or, say, statistical non-equilibrium thermodynamics, the process of averaging has to be understood in a somewhat pragmatical sense as there does not exist, at least at the moment, such a stringent and general statistical framework as in, say, Gibbsian statistical mechanics. In defining averaged field variables one has to resort to “sufficiently local” averages over, on the other side, “sufficiently many” microscopic degrees of freedom. This applies in particular to averaging over clock time intervals which have to be “*macroscopically small*” but sufficiently large as compared to the fundamental clock time interval, τ .

Assumption 4.3 (Statistical Version) *With the help of the above definitions we can formulate our assumption about the initial phase as follows: QX_0 is assumed to be characterized by*

$$\langle v(n) \rangle_s \text{ or } \langle v(n) \rangle_{st} \text{ "near" } |V| - 1 \quad (33)$$

or stated differently

$$\langle |E| \rangle_t \text{ "near" } \binom{|V|}{2} \quad (34)$$

Evidently there exist certain relations between these notions.

Observation 4.4 *With*

$$\langle v(n) \rangle_s := \sum_V v(n)/|V| \text{ and } \sum_V v(n)/2 = |E| \quad (35)$$

we have

$$\langle v(n) \rangle_s \cdot |V|/2 = |E| \quad (36)$$

and

$$\langle v(n) \rangle_{st} \cdot |V|/2 = \langle E \rangle_t \quad (37)$$

Observation/Definition 4.5 *On the level of the full network phase space (as compared with the underlying graph phase space) there are more quantities of physical significance, the statistical averages of which describe the details of the respective phases the microstate of the network is traversing. A case in point is the behavior of the local node states. We define*

$$\langle s \rangle_\nu, \text{ with } \nu := s, st \text{ or } \langle s_i \rangle_t \quad (38)$$

$$\Delta s := \langle (s - \langle s \rangle)^2 \rangle^{1/2} \quad \Delta s_{ik} := \langle (s_i - s_k)^2 \rangle^{1/2} \quad (39)$$

$$\langle (s_i(t + \tau) - s_i(t))^2 \rangle^{1/2} \text{ etc.} \quad (40)$$

with the averaging over space, time or space-time respectively.

A particularly important role concerning the dynamics of the wiring diagram is played by Δs_{ik} which, as we expect, behaves similar to $\langle |s_i - s_k| \rangle$. If this quantity is typically larger than the upper critical parameter of our model dynamics introduced in section 2 in a certain region of $QX(t)$, then many of the respective bonds will become temporarily inactive, i.e. are deleted in $G(t + \tau)$. We plan to

make a more detailed statistical or probabilistic analysis of our model network elsewhere, as such a full scale probabilistic analysis makes it necessary to investigate quite a few intricate and technical aspects as e.g. correlation lengths, the validity of various probabilistic tools in this context like limit theorems (in the case where the random variables are not completely independent) and the like.

For the time being, we will be contented with the following brief analysis. In our network and in particular in the initial phase, QX_0 , the average vertex degree $\langle v(n) \rangle$ is expected to be extremely high. Assuming that in the chaotic initial phase the bond variables, J_{ik} , being incident with an arbitrary node, n_i , are *statistically independent* to a large degree, we can infer the same for the node states, say, s_i, s_k .

Observation 4.6 *Assuming the almost statistical independence of bond variables being incident with n_i or n_k , we can conclude that (e.g. for the spacial average)*

$$\Delta s = O(\langle v(n) \rangle^{1/2}) = O(|V|^{1/2}) \quad (41)$$

in the initial phase QX_0 . As for the spacial averaging we have trivially $\langle s_i \rangle = \langle s_k \rangle$ this yields

$$\Delta s_{ik}^2 \approx \langle s_i \rangle^2 + \langle s_k \rangle^2, \quad (42)$$

that is, Δs_{ik} fluctuates roughly at the same order

$$\Delta s_{ik} = O(|V|^{1/2}) \quad (43)$$

in the phase QX_0 .

Remark 4.7 (Almost Statistical Independence) *As our network is, for the time being, a strictly deterministic system, an assumption like complete statistical independence in the above argument would be a little bit crude (see also the remarks made in the Résumé 3.2) as there may always exist certain (possibly short-lived) correlations between near by degrees of freedom so that something like a strict central limit theorem (see e.g. [28]) does not hold. But for the above argument to hold, much weaker variants would already be sufficient. We gave a preliminary discussion of this interesting point in section 3 of [29].*

Conclusion 4.8 *The above argument shows that it is plausible to expect huge fluctuations on very short scales in the densely connected initial phase QX_0 . If one takes our suggested model of a dynamical network law (with the ‘hysteresis interval (λ_1, λ_2) ’ appropriately tuned) one may entertain the speculative idea that under certain favorable circumstances an ‘avalanche’ may be set off in the course of which active bonds are constantly thinned out on an increasingly large scale. With this process fairly well developed the network may then enter a new geometric phase carrying an emergent superstructure we want to associate with the discrete statistical substratum of our continuous space-time.*

Corollary 4.9 *We think it is not too far fetched to relate this hypothetical process with the primordial (pre) big bang scenario, as we will show that our unfolding process is accompanied by an corresponding increase of macroscopic distance among the lumps of the emerging super structure in our network. Furthermore, if this picture proves to be correct, it would imply that in our expanding universe the infinitesimal physical lumps, representing the physical points, are constantly changing both their internal structure and external relationships (in a way which may however be very slow in the present state of the universe). This will be the central issue of the next section.*

Remark: As our main theme is the description and analysis of this new geometric phase, which is an ambitious task in its own right, we refrain at this place from going into further details concerning sufficient or necessary conditions which may trigger this hypothetical phase transition. In any case we are prepared to change the details of our model system if this turns out to be necessary (e.g. introduce bond-bond interactions, to give an example).

5 The Emergence of Space-Time

In this core section of our investigation we are going to describe how the presumed underlying (discrete) fine structure of our continuous space-time may look like. We want however to emphasize at this place that, while most of the details and observations we will present are rigorously proved, the overall picture is still only hypothetical. That is, we are able to describe a, as we think, fairly interesting scenario in quite some detail but are not yet able to show convincingly that our network actually evolves into exactly such a phase we are going to expound in the following.

In section 3 of this paper and in ([3], section 3.2) or section 4 of [29] we dealt with various properties of (maximal) subsimplices, $(m)ss$, (also called (maximal) complete subgraphs or cliques). In [3] we concentrated on their potential geometric content, in section 3 of this paper we discussed them in the context of random graphs. Both aspects will now be amalgamated when we introduce the concept of ‘*physical points*’ and their ‘*(causal) entanglement*’.

To begin with, we describe in broad outline our idea of the underlying discrete substratum of space-time.

The Qualitative Picture 5.1 (Physical Points)

1. *Physical points have a (presumably rich) internal structure, i.e. they consist of a (presumably) large number of nodes and bonds. In the words of Menger they are lumps.*
2. *We suppose that, what we are used to describe as fields at a space-time point (in fact, rather distributions in e.g. quantum field theory), are really internal excitations of these lumps.*

3. *In order to have a qualitative measure to tell these various physical points apart, that is, to discern what happens within a certain point or between different points, we conjecture that the individual physical points are particularly densely connected subgraphs of our network or graph. This then led to our interest in maximal subsimplices.*
4. *Typically (i.e. if a certain fraction of bonds has been eliminated), some of these lumps overlap with each other in a stronger or weaker sense, forming so to say ‘local groups’ while other will cease to overlap. This will then establish, in our picture, a kind of ‘proto-causality’ in our ‘proto-space-time’ and will be the central theme of our analysis.*

We will proceed by compiling a couple of simple observations concerning $(m)ss$.

Observation 5.2

1. *If the node degree, v_i , of n_i is smaller than ∞ then n_i can lie in at most a finite set of different simplices, an upper bound being provided by the number of different subsets of bonds emerging from n_i , that is 2^{v_i} .*
2. *The set of subsimplices is evidently ‘partially ordered’ by inclusion*
3. *Furthermore, if S is a simplex, each of its subsets is again a simplex (called a ‘face’)*
4. *It follows that each of the ‘chains’ of linearly ordered simplices (containing a certain fixed node) is finite. The corresponding length can be calculated in a similar way as in item 1 by selecting chains of sets of bonds, ordered by inclusion. In other words each chain has a maximal element. By the same token each node lies in at least one (but generically several) mss*
5. *A mss with n_i being a member can comprise at most $(v_i + 1)$ nodes, in other words, its order is bounded by the minimum of these numbers when n_i varies over the mss .*

Proof of item 1: Assume that S_k, S_l are two different simplices containing n_i . By definition n_i is linked with all the other nodes in S_k or S_l . As these sets are different by assumption, the corresponding subsets of bonds emerging from n_i are different. On the other side, not every subset of such bonds corresponds to a simplex (there respective endpoints need not form a simplex), which proves the upper bound stated above. \square .

Observation 5.3 *The class of simplices, in particular the mss , containing a certain fixed node, n_i , can be generated in a completely algorithmic way, starting from n_i . The first level consists of the bonds with n_i an end node, the second level*

comprises the triples of nodes ('triangles'), $(n_i n_k n_l)$, with the nodes linked with each other and so forth. Each level set can be constructed from the preceding one and the process stops when a *mss* is reached.

Remark: Note that at each intermediate step, i.e. having already constructed a certain particular subsimplex, one has in general several possibilities to proceed. On the other hand, a chain of such choices may differ at certain places from another one but may lead in the end to the same final simplex (in other words, being simply a permutation of the nodes of the former simplex).

Denoting the $(m)ss$ under discussion by capital S with certain indices or labels attached to it, this process can be pictorially abbreviated as follows:

$$S(n_0 \rightarrow \cdots \rightarrow n_k) \quad (44)$$

With $S(n_0 \rightarrow \cdots \rightarrow n_k)$ given, each permutation will yield the same *mss*, i.e:

$$S(n_0 \rightarrow \cdots \rightarrow n_k) = S(n_{\pi(0)} \rightarrow \cdots \rightarrow n_{\pi(k)}) \quad (45)$$

Furthermore each *mss* can be constructed in this way, starting from one of its nodes. Evidently this could be done for each node and for all possible alternatives as to the choice of the next node in the above sequence.

Definition 5.4 Let G_ν be a class of subgraphs of G .

1. $\cap G_\nu$ is the graph with $n \in V_{\cap G_\nu}$ if $n \in$ every V_{G_ν} ,
 $b_{ik} \in E_{\cap G_\nu}$ if $b_{ik} \in$ every E_{G_ν}
2. $\cup G_\nu$ is the graph with $n \in V_{\cup G_\nu}$ if $n \in V_{G_\nu}$ for at least one ν
 $b_{ik} \in E_{\cup G_\nu}$ if $b_{ik} \in E_{G_\nu}$ for at least one ν .

As every node or bond belongs to at least one *mss* (as can be easily inferred from the above algorithmic construction), we have

Corollary 5.5

$$\cup S_\nu = G \quad (46)$$

After these preliminary remarks we now turn to our main task, that is, the analysis of the web of these *mss* as the elementary building blocks of the next higher level of organisation.

In its surmised transition from the almost maximally connected and chaotic initial phase to the fully developed phase, QX/ST (i.e. QX plus superstructure ST), the underlying graph passes through several clearly distinguishable epochs.

We begin with the epoch where only a small fraction of bonds is shut off. Let us e.g. assume that α bonds with

$$1 \ll \alpha < n/2 \ll n(n-1)/2 \quad \text{for } n \text{ large} \quad (47)$$

are temporarily dead with n the order of the graph or network (i.e. the number of nodes) and the rhs of the above equation the maximal possible number of bonds. In other words, the network is supposed to be still near the initial phase. We observe that α arbitrarily selected bonds can at most connect $k \leq 2\alpha$ different nodes, hence there still exist at least $(n-k)$ nodes which are maximally connected, viz. they are spanning a still huge subsimplex $S' \subset G$. On the other hand there are at most $k \leq 2\alpha$ nodes with one or more incident bonds missing in the corresponding induced subgraph.

V_G can hence be split in the following way:

$$V_G = V_{S'} \cup V_N \quad (48)$$

with V_N the unique set of nodes with some of the α bonds among them missing, $V_{S'}$ the set of remaining nodes being maximally connected (by construction); i.e. they form a *ss*.

$$|V_N| = k \leq 2\alpha \quad (49)$$

Definition 5.6 $[\cup G_i]$ is the induced subgraph spanned by the nodes occurring in $\cup V_i$. Note that in general $[\cup G_i] \supset \cup G_i$, that is, it may rather be called its ‘closure’.

Observation 5.7

1. The simplex S' is contained in each of the occurring *mss*, S_ν , i.e.:

$$S' \subset \cap S_\nu \text{ and it holds a fortiori } S' = \cap S_\nu \quad (50)$$

2. Note that S' itself is never maximal since $[S' \cup n_i]$ is always a larger simplex with $n_i \in N$ and $[S' \cup n_i]$ being the induced subgraph spanned by $V_{S'}$ and n_i .
3. To each maximal simplex $S_\nu \subset G$ belongs a unique maximal subsimplex $N_\nu \subset N$ with

$$S_\nu = [S' \cup N_\nu] \quad (51)$$

4. It is important for what follows that S' can be uniquely characterized, without actually knowing the S_ν , by the following two properties: i) S' is a ss so that all bonds connecting nodes from $V_{S'}$ with $V - V_{S'}$ are “on”. ii) S' is maximal in this class of ss , that is, each node in $V - V_{S'}$ has at least one bond missing with respect to the other nodes in $V - V_{S'}$. An induced subgraph in G , having these properties is automatically the uniquely given S' !

Corollary 5.8 *From the maximality of the N_ν follows a general structure relation for the $\{S_\nu\}$ and $\{N_\nu\}$:*

$$\nu \neq \mu \rightarrow S_\nu \neq S_\mu \rightarrow N_\nu \neq N_\mu \quad (52)$$

and neither

$$N_\nu \subset N_\mu \text{ nor } N_\mu \subset N_\nu \quad (53)$$

viz. there always exists at least one $n_\nu \in V_{N_\nu}$ s.t. $n_\nu \notin V_{N_\mu}$ and vice versa.

Proof of the above observation:

1. Starting from an arbitrary node $n \in G$, it is by definition connected with all the nodes in S' , since if say n, n' are not connected they both belong to N (by definition). I.e., irrespectively how we will proceed in the construction of some S_ν , S' can always be added at any intermediate step, hence $S' \subset \cap S_\nu$. On the other side assume that $n \in \cap S_\nu$. This implies that n is connected with each node in $\cup S_\nu$. We showed above that $\cup S_\nu = G$, hence n is connected with all the other nodes, i.e. it is not in N , that is, $n \in S'$, which proves the statement.
2. As $n \in N$ is connected with each $n' \in S'$ (by definition of N and S'), the subgraph $[S' \cup n]$ is again a (larger) simplex.
3. We have $S' \subset S_\nu$ for all ν , hence

$$S_\nu \neq S_\mu \text{ implies } N_\nu \neq N_\mu \quad (54)$$

with $N_{\nu,\mu}$ the corresponding subgraphs in N .

With S_ν being a simplex, N_ν is again a subsimplex which is maximal in N . Otherwise S_ν would not be maximal in G .

On the other side each $S_\nu = [S' \cup N_\nu]$ is uniquely given by a maximal N_ν in N as each node in N is connected with all the nodes in S' .

□

We see from the above that as long as α , the number of dead (missing) bonds, is sufficiently small, i.e. $2\alpha < n$, there does exist an overlap S' , among the class of mss , S_ν . This overlap will become smaller as α increases with clock time t ; by the same token the number of mss will increase for a certain range of the parameter α while the respective size of the mss will shrink.. The above unique characterization of S' in item 4 makes it possible to attack the problem of the order of S' within the framework of random graphs in a more quantitative manner. Given a member G of $\mathcal{G}(n, p)$, S' is fixed by item 4 of the above observation. We are interested in the probability of S' having, say, r nodes.

The strategy is, as usual, to try to express the probability of such a configuration within G as the product of certain more elementary and (if possible) independent probabilities. Unfortunately this turns out to be relatively intricate in the above case and we are, at the moment, only able to provide certain upper and lower bounds for the probability under discussion. As this example shows that such questions may not always have simple and straightforward answers, it is perhaps worthwhile to dwell a little bit on this point.

The typical difficulties one usually encounters in this context are the following: The structure of the set of graphs in $\mathcal{G}(n, p)$ having a prescribed property may be rather complicated, so that it is difficult to avoid multiple counting of members when trying to calculate the order of such a set. A frequent reason for this is the intricate entanglement of the various pieces of a complicated graph, a case in point being the above description of S' in G . In our case the peculiar entanglement can be seen as follows.

Selecting r arbitrary vertices, the probability that the corresponding induced subgraph forms a ss is $p^{\binom{r}{2}}$ (see section 3). If this subgraph is to qualify as S' , i.e. $S' = \cap S_\nu, \cup S_\nu = G$, each of the nodes in N is connected with every node in S' . The probability for this property is $p^{r \cdot (n-r)}$. The difficult part of the reasoning concerns the subgraph N . We call the probability that N has just the structure being described above, $pr(N)$.

The following observation is helpful. As S' is unique in G , i.e. occurs only once, the corresponding random variable, $X_{S'}$, that we hit at such an S' of order r , when browsing through the set of induced r -subgraphs, is zero with at most one possible exception, that is, if S' has just the order r . Therefore the corresponding expectation value of $X_{S'}$ is, by the same token, also the probability of the property ' S'_r '.

Conclusion 5.9 *The probability that a random graph contains such a S' of order r is*

$$pr(S') = \binom{n}{r} \cdot p^{\binom{r}{2}} \cdot p^{r(n-r)} \cdot pr(N) \quad (55)$$

As far as we can see, it is not easy to disentangle $pr(N)$ into more elementary *independent* probabilities and master the complex combinatorics. Therefore we will, at the moment, only give (possibly crude) upper and lower bounds.

N , having $(n - r)$ nodes, is characterized as follows. Labeling the nodes from (1) to $(n - r)$, none of them is allowed to have the maximal possible degree (with respect to N), i.e. $(n - r - 1)$. The first step is simple. Starting with, say, node (1), the probability that at least one bond is missing is the complement of the probability that all possible bonds are present, i.e. $(1 - p^{(n-r-1)})$. The following steps will however become more and more cumbersome. Take e.g. node (2). In the above probability is already contained both the probability that either the bond b_{12} is missing or not. If b_{12} is not missing then some other bond b_{1i} has to be absent (by the definition of N). These two alternatives influence the possible choices being made at step two. In the former case the configuration where all bonds b_{2j} , $j > 2$ are present is admissible, in the latter case this possibility is forbidden. Depending of which choice we make at each step the algorithmic construction bifurcates in a somewhat involved manner. Evidently this first step yields a crude upper bound on $pr(N)$. Making at each step (i) the particular choice that there are always missing bonds among the bonds pointing to nodes (j) with $j > i$ provides a lower bound. We hence have.

Conclusion 5.10

$$(1 - p^{n-r-1}) \geq pr(N) \geq \prod_{j=1}^{n-r-1} (1 - p^{n-r-j}) = \prod_{j=1}^{n-r-1} (1 - p^j) \quad (56)$$

and for $pr(S' = \emptyset)$:

$$(1 - p^{n-1}) \geq pr(S' = \emptyset) \geq \prod_{j=1}^{n-1} (1 - p^j) \quad (57)$$

Observation 5.11 *The lower bound is interesting! Perhaps surprisingly, the occurring product is an important number theoretic function belonging to the field of partitions of natural numbers (see any good textbook about combinatorics as e.g. [30] or the famous book of Hardy and Wright, [31], the standard source being [32]). Our above random graph approach offers the opportunity to (re)derive and prove this number theoretic formula by purely probabilistic means, i.e. give it an underpinning which seems to be, at first glance, quite foreign. We will come back to this interesting point elsewhere.*

It is important to have effective estimates for the regime of probabilities, $p(n)$, so that S' is empty with a high probability. According to our philosophy this signals the end of the *embryonic epoch*, where all the supposed *protopoints* still overlap

(and hence are capable of direct interaction) and the beginning of the unfolding of a new phase with, as we hope, a more pronounced *near- and far-order* among the physical points.

Such an estimate can in fact be provided with the help of the above inequality. We have

$$pr(S' = \emptyset) > \prod_1^{n-1} (1 - p^j) > \prod_1^{\infty} (1 - p^j) = \sum_{k=0}^{\infty} a_k p^k \quad (58)$$

for $0 < p < 1$. The following (highly nontrivial) observation is due to Euler (cf. the above mentioned literature for more recent proofs):

Theorem 5.12

$$\prod_1^{\infty} (1 - p^j) = \sum_{k=0}^{\infty} (-1)^k \cdot \left(p^{\frac{1}{2}(3k^2+k)} + p^{\frac{1}{2}(3k^2-k)} \right) = 1 - p - p^2 + p^5 + p^7 \dots$$

$$\approx 1 - p - p^2 \quad (59)$$

for p small.

Conclusion 5.13 For p near zero, S' is empty with arbitrarily large probability $\lesssim 1$.

Remarks:

1. This shows that there exists in fact a regime of small p -values where the embryonic epoch no longer prevails. This holds the more so for an n -dependent p (which is very natural) and $p(n) \searrow 0$.
2. Note that there exists a possibly substantial class of bond configurations which have, up to now, been excluded in the above estimate, the inclusion of which would increase the relevant probability further.

One can provide other types of estimates concerning the overlap of $(m)ss$ which will further elucidate this point from other perspectives. It is particularly desirable to study the detailed structure of *entanglement* among the mss in the more developed epoch following the above embryonic scenario. We wil embark on such a program below, but before doing this, we want to briefly address two other aspects of the problem. The first one concerns the explicit construction of a large graph having exactly the properties we have been talking about. This is also worthwhile for its own sake as it is frequently rather difficult to construct large graphs with cetain prescribed properties.

The construction of the example goes as follows. Take $2k$ nodes, choose a subset G_1 consisting of exactly k nodes (n_1, \dots, n_k) , make G_1 a simplex. With

the remaining k nodes (n'_1, \dots, n'_k) we proceed in the same way, i.e. we now have two subsimplices G_1, G'_1 . We now choose a one-one-map from (n_1, \dots, n_k) to (n'_1, \dots, n'_k) , say:

$$n_i \rightarrow n'_i \quad (60)$$

We now connect all the n_i with the n'_j except for the k pairs (n_i, n'_i) . The graph G so constructed has

$$|E_G| = 2k(2k - 1)/2 - k = 2k(2k - 2)/2 \quad (61)$$

We see from this that, as in our above network scenario, the number of missing bonds is a relatively small fraction, hence, the example may be not so untypical.

We can now make the following sequence of observations:

Observation 5.14 1. $G_1 = S_1$ is already a mss as each $n'_i \in G'_1$ has one bond missing with respect to G_1 .

2. One gets new mss by exchanging exactly one n_i with its partner n'_i , pictorially:

$$[S_1 - n_i + n'_i] \quad (62)$$

yielding k further mss.

3. One can proceed by constructing another class of mss, now deleting (n_i, n_j) and adding their respective partners, i.e.:

$$[S_1 - n_i - n_j + n'_i + n'_j] \quad (63)$$

4. This can be done until we end up with the mss

$$[S_1 - n_1 - \dots - n_k + n'_1 + \dots + n'_k] = S'_1 \quad (64)$$

The combinatorics goes as follows:

$$|\{mss\}| = \sum_{\nu=0}^k \binom{k}{\nu} = (1 + 1)^k = 2^k \quad (65)$$

i.e., our $2k$ -node-graph (with k bonds missing) contains exactly 2^k mss of order k .

5. Evidently $S' = \emptyset$

Observation 5.15 1. We showed above that S' is non-empty as long as α , the number of missing bonds, is smaller than $n/2$, n the order of the graph G , since α bonds can at most connect 2α different nodes. On the other side, $\alpha = n/2$ implies

$$|E_G| = \binom{n}{2} - n/2 = n(n-2)/2 \quad (66)$$

or an average vertex degree

$$\langle v(n) \rangle_s = n - 2 \quad (67)$$

which is still very large.

2. The example constructed in the preceeding observation has

$$n = 2k, \alpha = k, S' = \emptyset \quad (68)$$

In other words, its parameters are just the “critical” ones regarding item 1.

3. In the light of these observations one may surmise that $\alpha = n/2$ is perhaps the threshold for $S' = \emptyset$ in the sense that, say,

$$\text{pr}(S' = \emptyset) = O(1) \quad (69)$$

for $\alpha \gtrsim n/2$.

On the other side, within the framework of random graphs, we have obtained the rigorous but presumably not optimal estimate

$$\text{pr}(S' = \emptyset) > \prod_1^\infty (1 - p^j) \approx 1 - p - p^2 \quad (70)$$

for p small. For a large graph $p = 1/2$ implies however

$$\alpha \approx \frac{1}{2} \binom{n}{2} \gg n/2 \quad (71)$$

That is, there is still a wide gap between these two values, a point which needs further clarification.

The other point deals with the *diameter* of a typical random graph and displays the at first glance perhaps surprising phenomenon that in a wide range of p -values, bounded away from zero, this diameter is only two!

Definition 5.16 (Diameter) The diameter, $\text{diam}(G)$, of a graph is defined as the greatest distance between two arbitrary nodes of the graph, i.e.

$$\text{diam}(G) = \max_{i,j} d(n_i, n_j) \quad (72)$$

Surprisingly, this diameter behaves rather uniformly over a wide range of p -values in a random graph. In order to derive a quantitative estimate we will proceed as follows. In a first step we calculate the probability that an arbitrary pair of nodes, (n_i, n_j) , is *not* directly connected with another arbitrary node, called x , by a pair of bonds. For a fixed pair n_i, n_j and x running through the set of remaining nodes this probability is (following the same line of reasoning as above) $(1 - p^2)^{(n-2)}$. There are $\binom{n}{2}$ such pairs, hence we have (calling the property under discussion A)

$$pr(A) = \binom{n}{2} \cdot (1 - p^2)^{(n-2)} \quad (73)$$

This has the following consequence:

Observation 5.17 *The probability that $diam(G)$ is two is $1 - pr(A)$.*

For n large, more precisely, $n \rightarrow \infty$, one can now calculate the n -dependent $p^*(n)$ -threshold so that $diam(G) = 2$ holds almost shurely for $p(n) > p^*(n)$ (cf. [24]). We content ourselves with a simple but nevertheless impressive result. With $p > 0$ fixed and $n \rightarrow \infty$ we have from the above estimate

$$pr(diam(G) = 2) \rightarrow 1 \quad (74)$$

Conclusion 5.18 *With $p > 0$ fixed and $n \rightarrow \infty$ $G(n, p)$ has diameter two almost shurely.*

In other words, if one wants to have graphs with large diameter, which, on the other side, do not fall apart in several pieces, one has to finetune $p(n) \rightarrow 0$ and/or confine the accessible phase space (dynamically), i.e. let the dynamics select a certain subset of graphs in $\mathcal{G}(n, p)$. This necessity or possibility has been discussed at length in the preceeding sections and is a feature, physicists are accustomed to. The result is even less dramatic as, according to our philosophy, it is rather the connectivity of lumps which interests us (that is, whole bunches of nodes) and which may be substantially lower (see below). Furthermore, such a web of node connections, extending over possibly large distances and lying below the web of lumps which, on its side, is expected to display a more pronounced near-/far-order, is actually wellcome as a (in our view) necessary prerequisite of quantum behavior!

Remark: Note that this result is not really surprising even from the purely mathematical point of view. A p bounded away from zero for $n \rightarrow \infty$ means a vertex degree scaling roughly as $p \cdot n$. This implies that on average each node is directly connected with a substantial fraction of all the nodes in the graph. The situation

is, on the other hand, completely different for, say, *lattice graphs* which frequently occur in ordinary physics. Here the node degree is low and constant and remains so for $n \rightarrow \infty$, implying that the diameter diverges.

In contrast to the above epoch we surmise the fully developed space-time superstructure of, say, the present epoch to be organized in a different way.

Conjecture 5.19 (Fully Developed Space-Time Picture)

1. *Each physical point or lump, identified in our framework with a particular mss, is surrounded by a local group of other lumps, overlapping with it. This represents its immediate (infinitesimal) neighborhood. Each of these mss, lying in the local group of, say, S_0 , is again surrounded by such a (slightly different) corona and so forth.*
2. *Note that the definition of mss does not exclude the possibility that there may exist still a substantial fraction of bonds, connecting two non-overlapping mss! Quite to the contrary, we consider this to be of tantamount importance as regards the incorporation of quantum interaction (and its, at least in our view, hidden non-locality or entanglement over appreciable macroscopic distances). On the other side, macroscopic distance is assumed to be expressed by the relative position of the lumps with respect to each other (see below).*
3. *The whole picture winds up to a two-story concept of space-time. For one there is its macroscopic or quasiclassical surface structure, i.e. ordinary almost classical space-time, consisting of the relatively smooth carpet of overlapping lumps. This granular texture defines, among other things, macroscopic distance, dimension and classical causality, i.e. (inter)action propagating through ordinary space-time, thus playing rather the role of a passive stage.*

For another, below this surface there does exist this intricate web of more or less randomly wired nodes with bonds, connecting lumps which may even be farther apart on the macroscopic scale. Our central conjecture is that this weaker and more stochastic communication between the various lumps is responsible for the quantum effects, we are observing in our present day effective quantum theories.

The following abbreviations are useful.

Definition 5.20 *For n_i, n_k (not being) connected by a bond we write*

$$n_i \sim n_k \quad (n_i \not\sim n_k) \tag{75}$$

We then have

Observation 5.21

1. $n_i \not\sim n_k$ implies that they are lying in different S_ν 's.
2. S_ν, S_μ are disjoint, i.e. $S_\nu \cap S_\mu = \emptyset$ iff

$$\forall n_\nu \in S_\nu \exists n_\mu \in S_\mu \text{ with } n_\nu \not\sim n_\mu \quad (76)$$

or vice versa.

Consequences 5.22 *This shows that it may well be that $S_\nu \cap S_\mu = \emptyset$ while the two mss have still a lot of ‘interbonds’, i.e. bonds connecting the one with the other. The guiding idea is however that V_{S_ν} and V_{S_μ} , as a whole, will typically be weaker entangled with each other than the nodes within S_ν or S_μ when the unfolding process is fully developed.*

Remark 5.23 *Note that the superstructure of physical points, i.e. $\{S_\nu\}$, imposes on G a structure called a ‘clique graph’ with the S_ν being the supernodes and two supernodes being adjacent if the corresponding cliques intersect. The ‘infinitesimal neighborhood’, $\mathcal{U}_1(S_0)$, of a point S_0 consists of the cliques having non-void intersection with S_0 .*

Definition 5.24 *With respect to the above clique graph we can speak of an*

1. ‘interior bond’ of a given S_ν , i.e:

$$b_{ik} \text{ with } n_i, n_k \in S_\nu \quad (77)$$

2. ‘exterior bond’ with respect to a given S_ν , i.e:

$$b_{ik} \text{ with } n_i, n_k \notin S_\nu \quad (78)$$

3. an ‘interbond’, i.e:

$$b_{ik} \text{ with } n_i \in S_\nu, n_k \in S_\mu, \nu \neq \mu \quad (79)$$

4. a ‘common bond’ of S_ν, S_μ if b_{ik} is an interior bond both of S_ν and S_μ .

5. a ‘true interbond’ b_{ik} if for $\nu \neq \mu$:

$$n_i \in S_\nu, n_k \in S_\mu, n_k \notin S_\nu \quad (80)$$

6. We then have the relation for given S_ν, S_μ :

$$\{\text{interbonds}\} - \{\text{common bonds}\} = \{\text{true interbonds}\} \quad (81)$$

Remark: The above relations between these classes of bonds describe the (time dependent) degree of entanglement among the S_ν , that is, among the physical proto points and, as a consequence, the physical near- and far-order on the level of ST , the macroscopic causality structure of space-time and the non-local entanglement we observe in quantum mechanics.

Observation 5.25 *We now have two (metric) structures on the network or graph, the original one with its neighborhood structure and distance function, $d(n_i, n_j)$, and the superstructure given by the clique graph and its coarse grained neighborhood structure of physical points and coarse grained distance function, $d_{cl}(S_i, S_j)$, which we regard as a protoform of our ordinary macroscopic distance. Note that there may exist a substantial number of interbonds on the lower level between supernodes S_i, S_j with $d_{cl}(S_i, S_j) \gg 1$.*

In the physics of many degrees of freedom what really matters, or gives “distance” a physical content, is not so much some abstract notion of distance but the strength of interaction or correlation between the various constituents. In most cases these characteristic quantities are in fact closely related with their spatial distance in the system under discussion and its (*intrinsic*) dimension. The latter one needs however not necessarily be the embedding dimension of ambient space (cf. e.g. [1])! Following this strand of ideas we will now organize our network.

Given two node sets A, B or the respective subgraphs we can count the number of bonds connecting them and regard this as a measure of their mutual dynamical coupling.

Definition 5.26 (Connectivity of Subgraphs) *With A, B being two node sets in a given graph, we denote by $|A \sim B|$ the actual number of bonds connecting the nodes of A with the nodes of B and by $|A \sim B|_m$ the maximal possible number. Then we call*

$$0 \leq c_{AB} := |A \sim B| / |A \sim B|_m \leq 1 \quad (82)$$

the ‘connectivity’ of the pair A, B . It represents the probability that a randomly chosen pair of nodes $n_A \in A, n_B \in B$ is connected by a bond. $|A \sim B|_m$ depends however on their relative position in G .

Observation 5.27

1.

$$A \cap B = \emptyset \rightarrow |A \sim B|_m = |A| \cdot |B| \quad (83)$$

($|A|, |B|$ the respective number of nodes), hence

$$c_{AB} = |A \sim B|/|A| \cdot |B| \quad (84)$$

2.

$$A = B \rightarrow |A \sim B|_m = \binom{|A|}{2} \quad (85)$$

3.

$$\begin{aligned} A \cap B \neq \emptyset \rightarrow |A \sim B|_m &= |(A - B) \sim (B - A)|_m \\ &+ |(A \Delta B) \sim (A \cap B)|_m + |(A \cap B) \sim (A \cap B)|_m \end{aligned} \quad (86)$$

i.e.:

$$|A \sim B|_m = |A - B| \cdot |B - A| + |A \cap B| \cdot (|A - B| + |B - A|) + \binom{|A \cap B|}{2} \quad (87)$$

with $A \Delta B$ being the symmetric difference of A and B .

Remark: There exist of course other possibilities to quantify the degree of mutual influence among the various regions of the graph. One could e.g. admit not only bonds but, say, paths up to a certain length leading from A to B . Furthermore there exist a variety of (not physically motivated) notions of connectedness in graph theory (e.g. the theorems of Menger, cf. e.g. [15]). The concept we developed above is adapted to our particular scenario with the mss as building blocks but with possibly a lot of surviving interbonds between disjoint grains S_ν, S_μ .

Conjecture 5.28 (The Metric/Topological Picture of QX/ST (a first Draft))

The picture we expect to emerge when the unfolding process is fully developed (i.e. α comparatively large) is now the following.

1. It is one of the many remarkable observations made in [24] that, perhaps against the usual intuition, random graphs tend to be surprisingly regular, i.e. the generic graph with, say, n nodes and m bonds tends to be almost 'translation invariant' in an averaged sense, viz. the node degree varies typically only within a small range (an effect of the peaked probability in phase space, similar to, say, related phenomena in statistical mechanics). We learned above that also the typical size of mss has such a peaked structure. It is perhaps natural to expect the same for their mutual entanglement.

2. Taking then a typical grain S_0 of ST , we expect its infinitesimal neighborhood $\{S_\nu^0\}$ to be densely connected with S_0 in the sense of the above definition. In other words:

$$c_{S_0 S_0} = 1 \quad , \quad c_{S_0 S_\nu^0} \lesssim 1 \quad (88)$$

3. Going on in this process of selecting grains, S_μ , in decreasing order with respect to $c_{S_0 S_\mu}$ we can construct shell after shell around S_0 with weaker and weaker connectivity as regards to the central element S_0 , i.e:

$$1 \geq c_{S_0 S_\nu^0} \geq c_{S_0 S_{\nu'}^1} \geq c_{S_0 S_{\nu''}^2} \geq \cdots \quad (89)$$

4. We expect (or hope) this process to be consistent with the neighborhood structure given on the supergraph ST which is defined by intersection of cliques. That is, we hope that node distance on ST corresponds more or less with the decrease in connectivity in the above sense.

So far we have described the overall scenario of the phase, QX/ST , in broad outline and have provided a certain arsenal of necessary conceptual tools, the most important being in our view the definition of physical points and their entanglement. What remains to be done is, on the one side, the development of a more detailed and quantitative underpinning of the still to some extent hypothetical picture. This concerns in particular the question which class of dynamical network laws is actually capable to drive the cellular network towards an attractor, having a structure as depicted by the phase, QX/ST . On the other hand, our general ansatz seems to be surprisingly rich so that it seems to be not too far fetched to expect possible regions of overlap to emerge in the future with respect to other fashionable approaches of a distinctly discrete flavor in quantum gravity.

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